Q.P. Code : 60861

# Second Semester M.Sc. Degree Examination, July 2019

(CBCS Scheme)

## Mathematics

## Paper M 201 T – ALGEBRA – II

Time : 3 Hours] [Max. Marks : 70

Instructions to Candidates :

1) *Answer any **FIVE** questions.*
2) *All questions carry equal marks.*

1. (a) Define :

(i) Nil radical N (A)

(ii) Jacobson radical J (A) of a ring A.

Prove that $x \in J(A)$ if and only if $1-xy$ is a unit in $A$ for all $x \in A$.

(b) Define the radical of an ideal $r(I)$ of a ring. For any ideals $I$ and $J$ of a ring $A$, prove that

(i) $I \subseteq J$ implies that $r(I) \subseteq r(J)$

(ii) $r(r(I)) = r(I)$

(c) Define extension and contraction of ideals with respect to a ring. Then show that

(i) $I \subseteq I^{ec}$ ; $J \supseteq J^{ce}$

(ii) $I^{e} = I^{ece}$ ; $J^{c} = J^{cec}$ (4 + 5 + 5)

2. (a) Show that every abelian group $G$ is a module over the ring of integers.

(b) Define a module homomorphism. State and prove the second isomorphism theorem for modules.

(c) Let $M$ be a finitely generated $A$-module and $I$ be an ideal of $A$ contained in a Jacobson radical $J(A)$ of a ring. Then show that $IM = M \Rightarrow M = O$.

(4 + 5 + 5)

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3. (a) Define a simple module. Show that an $A$- module $M$ is simple if and only if $M \cong A/I$ for some maximal ideal $I$ of $A$.

(b) Define an exact sequence. Prove that the sequence : $M' \xrightarrow{u} M \xrightarrow{u} M'' \to O$ is exact, if for all $A$-modules $N$, the sequence $O \to Hom(M'', N) \xrightarrow{u} Hom(M, N) \xrightarrow{u} Hom(M', N)$ is exact. **(7 + 7)**

4. (a) Show that a commutative ring with identity is Noetherean if and only if strictly ascending chain of ideals is of finite length.

(b) Define an Artinian ring. Show that every prime ideal is maximal in an artinian ring $A$.

(c) If $A$ is Noehterian ring, then prove that the polynomial ring $A[x]$ is Noetherian. **(5 + 4 + 5)**

5. (a) Define the degree of an extension $K$ of a field $F$. If $L$ is a finite extension of $K$ and $K$ is a finite extension of $F$, then prove that $L$ is a finite extension of $F$. Further, prove that $[L : F] = [L : K][K : F]$.

(b) Let $\mathbf{R}$ be the field of real numbers and $Q$ be the field of rationals. Prove the following :

(i) $a = \sqrt{2}$, $b = \sqrt{3}$ are algebraic over $Q$

(ii) $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$

(iii) $Q(\sqrt{2} + \sqrt{3})$ is an algebraic extension of of degree 4.

Determine the polynomial of degree 4 to verify that $\sqrt{2} + \sqrt{3}$ is algebraic of degree 4. **(6 + 8)**

6. (a) Prove that a polynomial of degree '$n$' over a field $F$ can have atmost '$n$' roots in any extension field.

(b) Determine the splitting field over the rationals of

(i) $f(x) = x^2 + \alpha x + \beta$

(ii) $f(x) = x^3 + \alpha x^2 + \beta x + \gamma$, $\alpha, \beta, \gamma \in Q$.

(c) Show that any splitting fields $E$ and $E'$ of the polynomial $f(x) \in F[x]$ and $f'(t) \in F'[t]$ respectively are isomorphic by an isomorphism $\phi$ with the property that $\alpha\phi = \alpha'$ for any $\alpha \in F$. **(4 + 5 + 5)**

7. (a) Prove that a regular pentagon is constructible (by using a straight edge and compass).

(b) Show that the polynomial $f(x) \in F[x]$ has multiple roots if and only if $f(x)$ and $f'(x)$ have a non-trivial common factors.

(c) Prove that any finite extension of a field $F$ of characteristic zero is a simple extension.

(4 + 5 + 5)

8. (a) Define a normal extension of a field. Prove that an extension $K$ of a field $F$ of degree two is normal.

(b) If $K$ is a normal extension of a field $F$ and if $L$ is an intermediate field of $K$ and F, then prove that $K$ is a normal extension of $L$ also.

(c) If $K$ is a finite extension of a field and if $G(K;F)$ is the group of all $F$ automorphisms of then prove that $G(K,F)$ is finite and $O(G(K,F)) \leq [K:F]$.

(4 + 6 + 6)

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